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Concerning Order Parameters in the Statistical Dynamical Theory of Diffraction

BY PIERRE J. BECKER* AND MOSTAFA AL HADDAD†

Laboratoire de Cristallographie, CNRS, and Université J. Fourier, 166X, 38042 Grenoble CEDEX, France

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Abstract

The development of the statistical dynamical theory of diffraction by real crystals involves two order parameters: a long-range and a short-range one. These parameters play a fundamental role in practical applications of the theory to real diffraction data. A reasonable probabilistic model is proposed to describe the phase correlation at two given scattering positions. This model allows for a tractable solution of the propagation equations for the beams.

Introduction

The propagation of X-rays or neutrons in real crystals can be described by Takagi's equations (Takagi, 1962, 1969; Kato, 1973). For a Bragg reflection associated with the reciprocal-lattice vector **h**, let D_0 and D_h be the amplitudes of the waves propagating in the incident direction (with coordinate s_0) and the diffracted direction (with coordinate s_h) respectively. Takagi's equations are

$$\begin{aligned} \partial D_0/\partial s_0 &= i\chi_{\bar{h}}\varphi D_h, \\ \partial D_h/\partial s_h &= i\chi_h\varphi^* D_0. \end{aligned} \tag{1}$$

In (1) χ_h is given by

$$\chi_h = (\lambda a C/V)F_h \tag{2}$$

where λ is the wavelength, V is the volume of the unit cell, $a = 10^{-12}$ cm for neutrons and $0{\cdot}28 \times 10^{-12}$ cm for X-rays, and C is the polarization factor for X-rays. F_h is the structure factor

$$|\chi_h| = 1/\Lambda$$

where Λ is the extinction distance.

The imperfect nature of the crystal occurs through the phase factor φ,

$$\varphi = \exp[2\pi i \mathbf{h}\,.\,\mathbf{u}(\mathbf{r})] \tag{3}$$

where $\mathbf{u}(\mathbf{r})$ is the local displacement from the perfect position $\mathbf{r}$.

Without a precise knowledge of the distortion field $[\mathbf{u}(\mathbf{r})]$, (1) can only be solved by introducing a statistical hypothesis concerning the distribution of $\mathbf{u}$ within the crystal. Kato (1980) proposed a statistical theory for describing the propagation of the beams in real crystals: this theory covers the whole range between perfect crystals (dynamical theory) and ideally imperfect crystals (kinematical theory). The beams, which are coherent for a perfect crystal, become partially incoherent, owing to phase couplings of the type $\varphi(\mathbf{r})\varphi^*(\mathbf{r}')$ which occur in the expression for the intensities. The statistical properties of the phase factor $\varphi(\mathbf{r})$ are thus essential for developing the theory.

Kato's theory has been modified by the present authors (Al Haddad & Becker, 1988) and then generalized (Becker & Al Haddad, 1989*a*, *b*). The purpose of the present paper is to discuss the statistical properties of $\varphi(\mathbf{r})$ and its spatial correlations, since this leads to the fundamental parameters appearing in the integrated reflectivity or in the extinction factor.

* Present address: Laboratoire de Minéralogie et Cristallographie, Université Pierre et Marie Curie, Tour 16, 4 place Jussieu, 75252 Paris CEDEX 05, France.

† Permanent address: Atomic Energy Commission, PO Box 6091, Damascus, Syria.

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Statistical hypothesis

(*A*) It is possible to consider $\mathbf{u}(\mathbf{r})$ as a random variable when $\mathbf{r}$ varies within the crystal. Let $p(\mathbf{u})$ be the probability distribution function for $\mathbf{u}$. Let E be the quantity

$$E = \int \exp(2\pi i \mathbf{h} \cdot \mathbf{u}) p(\mathbf{u})\, \mathrm{d}\mathbf{u}. \tag{4}$$

$p(\mathbf{u})$ is assumed to be an even function, so that the average $\mathbf{u}$ is zero. Owing to the definition of $p(\mathbf{u})$, one can also write

$$E = (1/V) \int_V \varphi(\mathbf{r})\, \mathrm{d}\mathbf{r}. \tag{5}$$

Kato (1980) defined E as the 'static Debye-Waller factor'. It is the long-range order parameter of the theory. $|E| \leq 1$, and $E = 1$ corresponds to the perfect crystal case.

Assuming t to be small compared with the crystal dimension, one can also define

$$F(\mathbf{t}) = (1/V) \int_V \varphi(\mathbf{r}) \varphi^*(\mathbf{r}+\mathbf{t})\, \mathrm{d}\mathbf{r}. \tag{6}$$

$F(\mathbf{t})$ is a spatial correlation function of the phase, which is assumed to be a real function.

Following Kato, we write

$$F(\mathbf{t}) = E^2 + (1 - E^2) g(\mathbf{t}) \tag{7}$$

where the pair-correlation function $g(\mathbf{t})$ is such that $g(0) = 1$.

Higher-order correlation functions can also be defined. However, the theories that have been developed so far are restricted to pair correlations. The meaning of this restriction is that the correlation length of $g(\mathbf{t})$ must be smaller than the average distance between two successive scattering events.

(*B*) The problem can be tackled from a different viewpoint. It is assumed in the diffraction theories that the beam intensities are uniquely defined when the distributions for $\varphi(\mathbf{r})$ and $\varphi(\mathbf{r})\varphi^*(\mathbf{r}')$ are given. We consider the ensemble of crystals (of a common shape) with the same distribution functions for $\varphi(\mathbf{r})$ and $\varphi(\mathbf{r})\varphi^*(\mathbf{r}')$. $u(\mathbf{r})$ and $\varphi(\mathbf{r})$ are assumed to be ergodic and homogeneous stochastic processes: their distribution over the various members of the ensemble, for fixed $\mathbf{r}$, is equivalent to their distribution over $\mathbf{r}$, for a fixed member of the ensemble. The observed diffracted intensity I_h can be viewed as the ensemble average

$$I_h = \langle |D_h|^2 \rangle. \tag{8}$$

Equation (4) can be written as

$$E = \langle \varphi(\mathbf{r}) \rangle. \tag{9}$$

E is independent of $\mathbf{r}$, owing to the property of homogeneity. Similarly

$$F(\mathbf{t}) = \langle \varphi(\mathbf{r}) \varphi^*(\mathbf{r}+\mathbf{t}) \rangle = E^2 + (1 - E^2) g(\mathbf{t}). \tag{10}$$

Because of homogeneity, $F(\mathbf{t})$ does not depend on $\mathbf{r}$.

For simplicity, isotropy will be assumed, so that $F(\mathbf{t})$ depends only on t. The correlation length τ is defined in any direction as

$$\tau = \int_0^\infty g(t)\, \mathrm{d}t \tag{11}$$

together with generalized correlation lengths

$$\tau_n = \int_0^\infty g^n(t)\, \mathrm{d}t. \tag{12}$$

(*C*) If $\hat{\mathbf{h}}$ is the unit vector along $\mathbf{h}$, we define

$$\xi(\mathbf{r}) = \hat{\mathbf{h}} \cdot \mathbf{u}(\mathbf{r}). \tag{13}$$

Let $P(\xi)$ be the probability distribution function for ξ,

$$E = \int P(\xi) \exp(2i\pi h \xi)\, \mathrm{d}\xi. \tag{14}$$

The joint probability for having ξ at $\mathbf{r}$ and ξ' at $(\mathbf{r}+\mathbf{t})$ is

$$P_2[\xi(\mathbf{r}), \xi'(\mathbf{r}+\mathbf{t})] = P_2(\xi, \xi', \mathbf{t}),$$

such that

$$\int \mathrm{d}\xi'\, P_2(\xi, \xi', \mathbf{t}) = P(\xi). \tag{15}$$

The correlation function $F(\mathbf{t})$ is thus

$$F(\mathbf{t}) = \int\int \exp[2\pi i h(\xi - \xi')] P_2(\xi, \xi', \mathbf{t})\, \mathrm{d}\xi\, \mathrm{d}\xi'. \tag{16}$$

(*D*) The optical path from a source S to a point M is an ordered trajectory, made of a sequence of segments parallel to the incident- and the diffracted-beam directions. If we consider two points M and M' in the crystal, one of them is always a preceding one with respect to the propagation process. Let us consider an ordered sequence of points $M_1, M_2, \ldots, M_n$. We can define the conditional probability of having a distortion ξ_n at $\mathbf{r}_n$, knowing the distortions $\xi_1(\mathbf{r}_1), \ldots, \xi_{n-1}(\mathbf{r}_{n-1})$, as

$$P_1[\xi_n(\mathbf{r}_n)/\xi_{n-1}(\mathbf{r}_{n-1}), \ldots, \xi_1(\mathbf{r}_1)].$$

The stochastic process $\xi(\mathbf{r})$ is said to be a Markov process (Feller, 1970; Van Kampen, 1981; Ziman, 1979) if this conditional probability depends only on $\xi_{n-1}(\mathbf{r}_{n-1})$ and not on the preceding events. This assumption is clearly consistent with the pair-correlation restriction: one considers only the correlation between nearest neighbours. Thus,

$$P_1[\xi_n(\mathbf{r}_n)/\xi_{n-1}(\mathbf{r}_{n-1}), \ldots, \xi_1(\mathbf{r}_1)] = P_1[\xi_n(\mathbf{r}_n)/\xi_{n-1}(\mathbf{r}_{n-1})]. \tag{17}$$

From (17) and elementary probability theory, we obtain

$$P_2(\xi_1, \xi_2, \mathbf{t}) = P(\xi_1) P_1[\xi_2(\mathbf{r}+\mathbf{t})/\xi_1(\mathbf{t})]. \tag{18}$$

If we consider a three-point event $[\xi_1(\mathbf{r}_1), \xi_2(\mathbf{r}_2), \xi_2(\mathbf{r}_3)]$, where $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is an ordered triplet of points,

then we obtain

$$P_3[\xi_3(\mathbf{r}_3), \xi_2(\mathbf{r}_2), \xi_1(\mathbf{r}_1)] = P(\xi_1) P_1[\xi_2(\mathbf{r}_2)/\xi_1(\mathbf{r}_1)] \times P_1[\xi_3(\mathbf{r}_3)/\xi_2(\mathbf{r}_2)]. \quad (19)$$

Integrating (19) over ξ_2, one finds

$$P_2[\xi_3(\mathbf{r}_3), \xi_1(\mathbf{r}_1)] = P(\xi_1) \int P_1[\xi_2(\mathbf{r}_2)/\xi_1(\mathbf{r}_1)] \times P_1[\xi_3(\mathbf{r}_3)/\xi_2(\mathbf{r}_2)]\, \mathrm{d}\xi_2.$$

Comparison with (17) finally leads to the Chapman–Kolmogorov equation

$$P_1[\xi_3(\mathbf{r}_3)/\xi_1(\mathbf{r}_1)] = \int P_1[\xi_3(\mathbf{r}_3)/\xi_2(\mathbf{r}_2)] \times P_1[\xi_2(\mathbf{r}_2)/\xi_1(\mathbf{r}_1)]\, \mathrm{d}\xi_2. \quad (20)$$

If $\xi(\mathbf{r})$ is a Markov process, (20) must be fulfilled.

Crystals with a high degree of perfection

We first consider the limiting case of a crystal with a high degree of perfection. This means that E is very close to 1. In other words, the random variable $\xi(\mathbf{r})$ will remain very small. Since dynamical effects occur only at small Bragg angles, this corresponds to the case where

$$h\xi(\mathbf{r}) = \mathbf{h} \cdot \mathbf{u}(\mathbf{r}) \ll 1.$$

In this case, one can make the approximation

$$F(\mathbf{t}) \sim 1 - \frac{1}{2!} 4\pi^2 h^2 \langle (\xi - \xi')^2 \rangle = 1 - 4\pi^2 h^2 \langle \xi^2 \rangle + 4\pi^2 h^2 \langle \xi \xi' \rangle.$$

Since E^2 can also be approximated as

$$E^2 \sim 1 - 4\pi^2 h^2 \langle \xi^2 \rangle,$$

$F(\mathbf{t})$ can be written in the form

$$F(\mathbf{t}) = E^2 + (1 - E^2) \langle \xi(\mathbf{r}) \xi(\mathbf{r}+\mathbf{t}) \rangle / \langle \xi^2 \rangle. \quad (21)$$

The usual correlation factor in probability theory is

$$\Gamma(t) = \langle \xi(\mathbf{r}) \xi(\mathbf{r}+\mathbf{t}) \rangle / \langle \xi^2 \rangle. \quad (22)$$

Within the present approximation, we see from (7) that the pair-correlation function $g(\mathbf{t})$, which is the leading term in the diffraction theory, can be represented as

$$\lim_{E \to 1} g(\mathbf{t}) = \Gamma(t). \quad (23)$$

Since all directions are supposed to be equivalent, $g(\mathbf{t})$ will not depend on $\mathbf{h}$ or on the Bragg angle. The correlation length τ will be constant.

The Gaussian random model

In order to deal with smaller values of E, or larger Bragg angles, it is necessary to make some statistical assumptions. The most common model in probability theory is the Gaussian distribution (Feller, 1970; Ziman, 1979),

$$P_2(\xi, \xi', \mathbf{t}) = \{2\pi \langle \xi^2 \rangle [1 - \Gamma^2(\mathbf{t})]\}^{-1} \times \exp\left(-\left\{ \frac{\xi^2 + \xi'^2 - 2\xi\xi'\Gamma(\mathbf{t})}{2\langle \xi^2 \rangle [1 - \Gamma^2(\mathbf{t})]} \right\} \right). \quad (24)$$

This model is justified when ξ can be considered as the sum of a large number of independent random contributions. For example, we may represent $\xi(\mathbf{r})$ as a Fourier sum,

$$\xi(\mathbf{r}) = \sum_{\mathbf{q}} a(\mathbf{q}) \exp(2i\pi \mathbf{q} \cdot \mathbf{r}). \quad (25)$$

When $\mathbf{q}$ varies in the Brillouin zone, and if one assumes statistical independence among the various components $a(\mathbf{q})$, then (24) follows from the central limit theorem. $F(\mathbf{t})$ takes the simple form

$$F(\mathbf{t}) = \exp\{-4\pi^2 h^2 \langle \xi^2 \rangle [1 - \Gamma(\mathbf{t})]\} \quad (26)$$

and E is given by

$$E = \exp\{-2\pi^2 h^2 \langle \xi^2 \rangle) = \exp[-(2\pi^2/3) h^2 \langle u^2 \rangle]. \quad (27)$$

If it is further assumed that $\xi(\mathbf{r})$ is a Markov process, the application of (20) leads to

$$\Gamma(\mathbf{t}+\mathbf{t}') = \Gamma(\mathbf{t}) \Gamma(\mathbf{t}') \quad (28)$$

where

$$\mathbf{t} = x\hat{\mathbf{u}}_0 + y\hat{\mathbf{u}}_h \quad x, y \geq 0$$

($\hat{\mathbf{u}}_0$ and $\hat{\mathbf{u}}_h$ are unit vectors along the incident and diffracted directions). If all directions are equivalent, (28) is only compatible with

$$\Gamma(\mathbf{t}) = \exp(-x/\tau) \exp(-y/\tau), \quad (29)$$

τ being a characteristic length, which measures the domains of phase coherence.

The advantage of the decomposition (7) for $F(\mathbf{t})$, in the development of the diffraction theory, is the possibility of separating the intensities into coherent and incoherent contributions. However, we notice that $F(\mathbf{t})$, as given by (26), is not well suited for the decomposition (7), since $g(\mathbf{t})$ would be a complicated function depending on $\mathbf{h}$.

However, we may notice that the Gaussian model might permit another type of solution of Takagi's equations. Using the notation of Becker & Al Haddad (1989*a*, *b*), (1) can be written in the integral form

$$D_h(s_0, s_h) = -\chi^2 \int_0^{s_0} \mathrm{d}\xi \int_0^{s_h} \mathrm{d}\eta\, \varphi^*(s_0, \eta) \varphi(\xi, \eta) D_h(\xi, \eta) + i\chi\varphi^*(s_0, 0) = -\hat{L} D_h + i\chi\varphi^*(s_0, 0), \quad (30)$$

$$D_h = \sum_{n=0}^{\infty} (-1)^n \hat{L}^n [i\chi\varphi^*(s_0, 0)] \quad (31)$$

and

$$I_h = \langle |D_h|^2 \rangle = \sum_{n,m} (-1)^{n+m} < \hat{L}^n [i\chi\varphi^*(s_0, 0)] \hat{L}^{*m} [i\chi\varphi(s_0, 0)]. \quad (32)$$

Each term in (32) would involve integrals over Gaussian-type correlations, each of the form (26). Such integrations are in principle possible, but would involve exponential integral functions. The notion of incoherent *versus* coherent processes would be lost.

A simple model for the pair distribution

Rather than assuming a Gaussian model, we wish to discuss some general properties of $P_2(\xi, \xi', \mathbf{t})$.

Where $\mathbf{t} \to \mathbf{0}$, $\xi'(\mathbf{r}+\mathbf{t}) \to \xi(\mathbf{r})$, for any model of continuous disorder (Ziman, 1979). This leads to

$$\lim_{\mathbf{t}\to\mathbf{0}} P_2[\xi, \xi', \mathbf{t}] = P(\xi)\delta(\xi' - \xi). \quad (33)$$

This is the complete correlation limit.

Where $\mathbf{t} \to \infty$, $\xi(\mathbf{r})$ and $\xi'(\mathbf{r}+\mathbf{t})$ are no longer correlated. Therefore

$$\lim_{\mathbf{t}\to\infty} P_2[\xi, \xi', \mathbf{t}] = P(\xi)P(\xi'). \quad (34)$$

This is the uncorrelated limit.

The simplest model is to assume a linear relation between these two limits, (33) and (34). We thus write

$$P_2(\xi, \xi', \mathbf{t}) = \alpha(\mathbf{t})P(\xi)\delta(\xi' - \xi) + [1 - \alpha(\mathbf{t})]P(\xi)P(\xi'), \quad (35)$$

where the weighting factor $\alpha(\mathbf{t})$ is to be determined. $\alpha(\mathbf{t})$ must fulfil the conditions

$$\alpha(0) = 1, \qquad \alpha(\infty) = 0. \quad (36)$$

Let us calculate the quantity

$$\langle \xi(\mathbf{r})\xi(\mathbf{r}+\mathbf{t}) \rangle = \int\int \xi\xi' P_2(\xi, \xi', \mathbf{t})\, d\xi\, d\xi'.$$

The application of (35) leads to

$$\langle \xi(\mathbf{r})\xi(\mathbf{r}+\mathbf{t}) \rangle = \langle \xi^2 \rangle \alpha(\mathbf{t}) + [1 - \alpha(\mathbf{t})]\langle \xi \rangle^2.$$

Since $\langle \xi \rangle$ is assumed to be zero,

$$\alpha(\mathbf{t}) = \Gamma(\mathbf{t}), \quad (37)$$

which gives a very clear meaning to the weighting factor $\alpha(\mathbf{t})$ in (35):

$$P_2(\xi, \xi', \mathbf{t}) = \Gamma(\mathbf{t})P(\xi)\delta(\xi' - \xi) + [1 - \Gamma(\mathbf{t})]P(\xi)P(\xi'). \quad (38)$$

Using (16), we get for $F(\mathbf{t})$

$$F(\mathbf{t}) = \Gamma(\mathbf{t}) + [1 - \Gamma(\mathbf{t})]E^2 = E^2 + [1 - E^2]\Gamma(\mathbf{t}), \quad (39)$$

from which we conclude that $g(\mathbf{t})$ is equal to $\Gamma(\mathbf{t})$.

Within this model, $g(\mathbf{t})$ is independent of $\mathbf{h}$. The model is indeed well suited for the decomposition (7) and thus for a separation into coherent and incoherent contributions to the intensity.

E is of course $\mathbf{h}$ dependent, and can reasonably well be represented as

$$E = \exp[-(2\pi^2/3)h^2\langle u^2 \rangle]. \quad (40)$$

If we further assume $\xi(\mathbf{r})$ to be a Markov process, we make use of the Chapman-Kolmogorov identity (20), with

$$P_1[\xi_2(\mathbf{r}_2)/\xi_1(\mathbf{r}_1)] = \Gamma(\mathbf{r}_2 - \mathbf{r}_1)\delta(\xi_2 - \xi_1) + [1 - \Gamma(\mathbf{r}_2 - \mathbf{r}_1)]P(\xi_2). \quad (41)$$

For an ordered triplet $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, one gets

$$\Gamma(\mathbf{r}_3 - \mathbf{r}_1)\delta(\xi_3 - \xi_1) + [1 - \Gamma(\mathbf{r}_3 - \mathbf{r}_1)]P(\xi_3) = \Gamma(\mathbf{r}_3 - \mathbf{r}_2)\Gamma(\mathbf{r}_2 - \mathbf{r}_1)\delta(\xi_3 - \xi_1) + [1 - \Gamma(\mathbf{r}_3 - \mathbf{r}_2)\Gamma(\mathbf{r}_2 - \mathbf{r}_1)]P(\xi_3),$$

which is fulfilled if

$$\Gamma(\mathbf{r}_3 - \mathbf{r}_1) = \Gamma(\mathbf{r}_3 - \mathbf{r}_2)\Gamma(\mathbf{r}_2 - \mathbf{r}_1).$$

We retrieve the condition (28), and thus

$$g(\mathbf{t}) = g(x)g(y), \qquad g(x) = \exp(-x/\tau), \quad (42)$$

τ being the coherence length of the phase factor φ.

We notice that (42) was implied in several approximations when developing the diffraction theory (Becker & Al Haddad, 1989*a*, *b*).

The present model is therefore consistent with the development of the theory for a real crystal. The last point to be considered is the limit of validity of the Markov process to describe multiple scattering. As we said earlier, it is equivalent to assuming that the correlation length is smaller than the average distance between points where multiple scattering occurs. The average distance between centres of scattering is of the order of Λ, the extinction length. Therefore, the present model may be considered as valid if

$$\tau \ll \Lambda, \quad (43)$$

again a condition that was implied in the theories.

Concluding remarks

It has been possible to describe phase coupling by a model which is a distance-weighted linear combination between a fully correlated and a totally uncorrelated model. This model turns out to be totally consistent with the assumptions that are to be made in the development of the diffraction theory. It leads to a short-range-order parameter τ that is a constant, E being represented by a Gaussian in $\mathbf{h}$. Two parameters are thus needed, $\langle u^2 \rangle$ and τ.

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A Simulated Annealing Approach to the Search Problem of Protein Crystallography

BY S. SUBBIAH* AND S. C. HARRISON†

Committee on Higher Degrees in Biophysics and Howard Hughes Medical Institute, 7 Divinty Avenue, Harvard University, Cambridge, MA 02138, *USA*

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Abstract

With the growing availability of computer power it has become routine to perform exhaustive multi-dimensional searches in protein crystallography. Specifically, in cases where homologous or partially homologous structures are available, the initial interpretation of poor electron density maps is done by performing computer-intensive rotational and translational searches in real space. Often such calculations of the best fit between structure and map cannot even be attempted owing to the vast computing effort involved (years of MicroVAX II time). Here, the combinatorial optimization method, simulated annealing, is shown to reduce substantially the computing effort involved and also to permit computations that are beyond the reach of current algorithms. This is illustrated with practical examples involving the structure determinations of the human histocompatibility antigen HLA-A2 and an influenza virus hemagglutinin-sialic acid complex.

1. Introduction

In cases where poor and not-readily interpretable electron density (e.d.) maps of protein structures are available, it has become common to perform real-space rotation and translation searches using homologous or partially homologous protein structures as search objects against the available map (Huber, 1965; Colman & Webster, 1985; Reynolds, Remington, Weaver, Fisher, Anderson, Ammon & Matthews, 1985). These exhaustive searches in general require large amounts of computer time and often such searches cannot be attempted for this reason. However, when attempted they typically require on the order of hundreds of MicroVAX II c.p.u. hours (Colman & Webster, 1985; Reynolds *et al.*, 1985) (*NB* a Cray I is on the order of 100 times faster than a MicroVAX II). Even in these cases it may be necessary to approximate the search object to the α-carbon backbone alone. Such exhaustive calculations ascertain the best-fit six-dimensional orientation of the search object in the map. However, depending on the appropriateness of the structural homologue and the 'noise' level of the map, the optimal solution may not necessarily correspond to the true solution of the crystallographic problem. Nonetheless, the correct solution is always found to be among the better solutions to the search procedure. In short, the problem is that of using the least-approximate search object and to obtain a list of good solutions to the search problem while minimizing the amount of computer effort.

The complex optimization technique of *simulated annealing* (also known as the Metropolis algorithm) is shown in this paper to have these desired qualities (Metropolis, Rosenbluth, Rosenbluth, Teller & Teller, 1953). This method, which is a variant of the Monte Carlo method, has been applied in recent years with much success to large optimization problems ranging from spin-glass theory in solid-state physics to the classic travelling salesman problem of computer science (Van Hemmen & Morgenstern, 1983; Kirkpatrick, Gelatt & Vecchi, 1983). It has also recently been found to be an effective tool in the

* Committee on Higher Degrees in Biophysics.

† Committee on Higher Degrees in Biophysics and Howard Hughes Medical Institute.

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